ISRAEL JOURNAL OF MATHEMATICS 79 (1992), 367–379

ON THE CLASS OF MEASURABLE CARDINALS WITHOUT THE AXIOM OF CHOICE

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ABSTRACT

Using techniques of Gitik in conjunction with a large cardinal hypothesis whose consistency strength is strictly in between that of a supercompact and an almost huge cardinal, we obtain the relative consistency of the theory "$ZF + \neg AC_\omega + \kappa > \omega$ is measurable iff κ is the successor of a singular cardinal".

It is well known that when the Axiom of Choice (AC) becomes false, the structure of the set theoretic universe can be radically altered. As an example, $\aleph_1$ can be singular [L]. Indeed, all uncountable successor and limit cardinals can be singular [G1], or in fact, any desired uncountable successor cardinal, along with all limit cardinals, can be singular [G2].

The bizarre behavior of the set theoretic universe in the absence of AC extends to large cardinals as well. Large cardinals such as Ramsey and measurable cardinals can be successor instead of limit cardinals [J1], [T], [A3], [A4], [A5], [AH]. It is even possible to have non-vacuously the consistency of the theory "$ZF + \neg AC_\omega + \kappa > \omega$ is regular iff κ is measurable" [A1].

The purpose of this paper is to show that yet more unusual possibilities for the structure of the class of measurable cardinals without the Axiom of Choice can occur. It was shown in [A4], [A5], and [AH] that it is possible to force and obtain a model in which the successor of a singular cardinal is measurable. (See [K] for a discussion of this result in the context of the Axiom of Determinacy (AD).) We generalize this result in the spirit of [A1] to show the consistency of

Received July 29, 1991 and in revised form October 17, 1991

the theory "$ZF + \neg AC_\omega + \kappa > \omega$ is measurable iff κ is the successor of a singular cardinal" relative to a certain large cardinal assumption. Specifically, we prove the following

THEOREM: *Let $V \models$ "$ZFC+$ There is a cardinal κ_0 so that:*

1. *κ_0 is superstrong via j, i.e., there is a $j\colon V \to M$ with $\mathrm{crit}(j) = \kappa_0$ so that $V_{j(\kappa_0)} \subseteq M$.*
2. *κ_0 is $< j(\kappa_0)$ supercompact, i.e., for all $\alpha < \kappa_0$, κ_0 is α supercompact.*
3. *The inner model M is so that $V_{j(\kappa_0^*+1)} \subseteq M$, where κ_0^* is the least cardinal $> \kappa_0$ which is $< j(\kappa_0)$ supercompact".*

Let, in addition, $A, B \subseteq \kappa_0$, $A, B \in V$ satisfy the following properties:

a. $A \cap B = \emptyset$ and $A \cup B = \kappa_0$.

b. *If $\lambda < \kappa_0$ is a limit ordinal, then $\lambda \in A$.*

c. *If $\nu \in A$, then $\nu + 1, \nu + 2 \in B$.*

There is then a sequence $\langle \alpha_\nu : \nu < \kappa_0 \rangle$ and a model N_A of height κ_0 for the theory "$ZF + \neg AC_\omega +$ The cardinal γ is singular iff $\gamma = \alpha_\nu$ for some $\nu \in A$ + For all $\nu \in A$, $\gamma^+ = \alpha_\nu^+$ is measurable and carries a normal measure + If γ is not the successor of a singular cardinal, then γ isn't measurable".

Note that the conditions on A and B imply that B is composed entirely of successor ordinals $< \kappa_0$ and that the final model N_A will satisfy "κ is measurable iff κ is the successor of a singular cardinal". Since $N_A \models \neg AC_\omega$, i.e., since $N_A \not\models DC$, the fact that each measurable cardinal carries a normal measure is significant. Note further that if κ is an almost huge cardinal, i.e., if there is a $j\colon V \to M$ with $\mathrm{crit}(j) = \kappa$ so that $M^{<j(\kappa)} \subseteq M$, then κ possesses properties (1)-(3) above, and as shown in [A2], κ has a normal measure concentrating on cardinals satisfying properties (1)-(3) above. Thus, the κ_0 used in the construction of the model N_A is strictly weaker in consistency strength than an almost huge cardinal (and since $\kappa_0^* > \kappa_0$ is inaccessible, is strictly stronger in consistency strength than a supercompact cardinal).

Turning now to our proof, the proof of the Theorem uses Gitik's techniques of [G2] (see also [A1], [A2], and [A3]) to construct N_A. To begin, if V satisfies the hypotheses of the Theorem, let A, B, κ_0, and $j\colon V \to M$ be as in these hypotheses. The first step in the proof is to define a Radin sequence of measures $\mu_{<\kappa_0^+} = \langle \mu_\alpha \colon \alpha < \kappa_0^+ \rangle$ of length κ_0^+ over $P_{\kappa_0}(\kappa_0^*)$. Specifically, if $\alpha = 0$, μ_α is defined by $X \in \mu_\alpha$ iff $\langle j(\beta)\colon \beta < \kappa_0^* \rangle \in j(X)$, and if $\alpha > 0$, μ_α is defined by

$X \in \mu_\alpha$ iff $\langle \mu_\beta \colon \beta < \alpha \rangle =_{\mathrm{df}} \mu_{<\alpha} \in j(X)$. As in [A2], properties (1) and (3) above ensure that this definition makes sense.

Next, using $\mu_{<\kappa_0^+}$, we let $R_{<\kappa_0^+}$ be supercompact Radin forcing defined over $V_{\kappa_0} \times P_{\kappa_0}(\kappa_0^*)$. The particulars of the definition can be found in [G2] and [A3]; however, in the interest of completeness and clarity, we repeat the definition here. $R_{<\kappa_0^+}$ is composed of all finite sequences of the form

$$\langle \langle p_0, u_0, C_0, \rangle, \ldots, \langle p_n, u_n, C_n \rangle, \langle \mu_{<\kappa_0^+}, C \rangle \rangle$$

satisfying the following properties:

1. For $0 \le i < j \le n$, $p_i \subsetneqq p_j$, where for $p, q \in P_{\kappa_0}(\kappa_0^*)$, $p \subsetneqq q$ means $p \subseteq q$ and $\overline{p} < q \cap \kappa$. ($\overline{p}$ is the order type of p.)
2. For $0 \le i \le n$, $p_i \cap \kappa_0$ is a $< \kappa_0$ supercompact cardinal.
3. $\overline{p_i}$ is the least cardinal $> p_i \cap \kappa_0$ which is a $< \kappa_0$ supercompact cardinal. In analogy to our earlier notation and the notation of [G2], we write $\overline{p_i} = (p_i \cap \kappa_0)^*$.
4. For $0 \le i \le n$, u_i is a Radin sequence of measures over $V_{p_i \cap \kappa_0} \times P_{p_i \cap \kappa_0}(\overline{p_i})$ where $(u_i)_0$, the 0^{th} coordinate of u_i, is a supercompact measure over $P_{p_i \cap \kappa_0}(\overline{p_i})$.
5. C_i is a sequence of measure 1 sets for u_i.
6. C is a sequence of measure 1 sets for $\mu_{<\kappa_0^+}$.
7. For each $p \in (C)_0$, where $(C)_0$ is the coordinate of C so that $(C)_0 \in \mu_0$, $\cup_{i=0}^n p_i \subsetneqq p$.
8. For each $p \in (C)_0$, $\overline{p} = (p \cap \kappa_0)^*$ and $p \cap \kappa_0$ is a $< \kappa_0$ supercompact cardinal.

Properties (4), (5), and (6) are all standard properties of Radin forcing. Properties (1), (2), (3), (7), and (8) all follow since properties (1)–(3) of κ_0 of the hypotheses of the Theorem imply $M \models$ "κ_0 is $< j(\kappa_0)$ supercompact and κ_0^* is the least cardinal $> \kappa_0$ which is $< j(\kappa_0)$ supercompact", so by reflection, $\{p \in P_{\kappa_0}(\kappa_0^*) \colon p \cap \kappa_0$ is a $< \kappa_0$ supercompact cardinal and $\overline{p}$ is the least $< \kappa_0$ supercompact cardinal $> p \cap \kappa_0\} \in \mu_0$.

We recall now the definition of the ordering on $R_{<\kappa_0^+}$. If

$$\pi_0 = \langle \langle p_0, u_0, C_0 \rangle, \ldots, \langle p_n, u_n, C_n \rangle, \langle \mu_{<\kappa_0^+}, C \rangle \rangle \text{ and}$$
$$\pi_1 = \langle \langle q_0, v_0, D_0 \rangle, \ldots, \langle q_m, v_m, D_m \rangle, \langle \mu_{\kappa_0^+}, D \rangle \rangle,$$

then π_1 extends π_0 iff the following conditions hold:

1. For each $\langle p_j, u_j, C_j\rangle$ which appears in π_0 there is a $\langle q_i, v_i, D_i\rangle$ which appears in π_1 so that $\langle q_i, v_i\rangle = \langle p_j, u_j\rangle$ and $D_i \subseteq C_j$, i.e., for each coordinate $(D_i)_\alpha$ and $(C_j)_\alpha$, $(D_i)_\alpha \subseteq (C_j)_\alpha$.
2. $D \subseteq C$.
3. $n \le m$.
4. If $\langle q_i, v_i, D_i\rangle$ does not appear in π_0, let $\langle p_j, u_j, C_j\rangle$ (or $\langle \mu_{<\kappa_0^+}, C\rangle$) be the first element of π_0 so that $p_j \cap \kappa_0 > q_i \cap \kappa_0$. Then
   a) q_i is order isomorphic to some $q \in (C_j)_0$.
   b) There exists an $\alpha < \alpha_0$, where α_0 is the length of u_j, so that v_i is isomorphic "in a natural way" to an ultrafilter sequence $v \in (C_j)_\alpha$.
   c) For β_0 the length of v_i, there is a function $f\colon \beta_0 \to \alpha_0$ so that for $\beta < \beta_0$, $(D_i)_\beta$ is a set of ultrafilter sequences so that for some subset $(D_i)_\beta{}'$ of $(C_j)_{f(\beta)}$, each ultrafilter sequence in $(D_i)_\beta$ is isomorphic "in a natural way" to an ultrafilter sequence in $(D_i)_\beta{}'$.

For further information on the definition of the ordering on $R_{<\kappa_0^+}$ (including the meaning of "in a natural way"), readers are referred to [A3] and [FW].

Before giving the definition of the partial ordering used in the construction of the model for our Theorem, we recall the definition of two key partial orderings. If $\alpha < \beta$ are regular cardinals, then $\mathrm{Col}(\alpha, < \beta)$ is just the usual Lévy collapse of all cardinals in the interval (α, β) to α, i.e., $\mathrm{Col}(\alpha, < \beta) = \{f\colon \alpha \times \beta \to \alpha : f$ is a function so that $|\mathrm{dom}(f)| < \alpha$ and $f(\langle \gamma, \delta\rangle) < \delta\}$ ordered by inclusion. For $\sigma \in (\alpha, \beta)$ a regular cardinal, $f \in \mathrm{Col}(\alpha, < \beta)$, $f|\sigma = \{\langle\langle \gamma, \delta\rangle, \rho\rangle \in f\colon \delta < \sigma\}$. If G is V-generic over $\mathrm{Col}(\alpha, < \beta)$, then $G|\sigma = \{f|\sigma\colon f \in G\}$ is V-generic over $\{f|\sigma\colon f \in \mathrm{Col}(\alpha, < \beta)\} = \mathrm{Col}(\alpha, < \beta)|\sigma = \mathrm{Col}(\alpha, < \sigma)$.

If α is β supercompact, then let $\mathcal{U}$ be a normal measure over $P_\alpha(\beta)$ satisfying the Menas partition property [M]. (Such an ultrafilter will always exist if α is 2^β supercompact, a restriction which will cause no problems since we will be working with cardinals $\alpha < \beta < \kappa_0$, so α can be chosen to be $< \kappa_0$ supercompact.) Supercompact Prikry forcing $SC(\alpha, \beta)$ is then defined as all sequences of the form $\langle p_0, \ldots, p_n, C\rangle$ so that:

1. $n \in \omega$ and $C \in \mathcal{U}$.
2. For $0 \le i \le n$, $p_i \in P_\alpha(\beta)$.
3. For $0 \le i < j \le n$, $p_i \subsetneqq p_j$.
4. For each $q \in C$, $p_n \subsetneqq q$.

For $\pi_1 = \langle p_0, \ldots, p_n, C\rangle$ and $\pi_2 = \langle q_0, \ldots, q_m, D\rangle$, π_2 extends π_1 iff:

1. $n \le m$.
2. For $0 \le i \le n$, $p_i = q_i$.
3. For $n + 1 \le i \le m$, $q_i \in C$.
4. $D \subseteq C$.

We now define a partial ordering P' by

$$P' = R_{<\kappa_0^+} \times \prod_{\{\langle \alpha,\beta\rangle:\ \alpha<\beta<\kappa_0 \text{ are regular cardinals}\}} \mathrm{Col}(\alpha, <\beta)$$

$$\times \prod_{\{\langle \alpha,\beta\rangle:\ \alpha<\beta<\kappa_0 \text{ is so that } \beta \text{ is a regular cardinal and } \alpha \text{ is } <\kappa_0 \text{ supercompact}\}} SC(\alpha,\beta)$$

ordered componentwise, and let P be the subordering of P' consisting of all conditions of finite support, also ordered componentwise. Let G be V-generic over P. The model N_A for A as in the statement of the Theorem will be a submodel of $V[G]$ similar to the models N_A of [G2] and [A3]. We describe this model in more detail below.

Let G_0 be the projection of G onto $R_{<\kappa_0^+}$. For any condition

$$\pi = \langle\langle p_0, u_0, C_0\rangle, \ldots, \langle p_n, u_n, C_n\rangle, \langle \mu_{<\kappa_0^+}, C\rangle\rangle \in R_{<\kappa_0^+}$$

or any condition $\pi = \langle p_0, \ldots, p_n, C\rangle \in SC(\alpha,\beta)$ call $\langle p_0, \ldots, p_n\rangle$ the p-part of π. Let $R = \{p\colon \exists\pi \in G_0[p \in \text{p-part}(\pi)]\}$ and let $R_l = \{p\colon p \in R \text{ and } p \text{ is a limit point of } R\}$. We define three sets E_0, E_1, and E_2 by $E_0 = \{\alpha\colon$ For some $\pi \in G_0$ and some $p \in$p-part(π), $p \cap \kappa_0 = \alpha\}$, $E_1 = \{\alpha\colon \alpha$ is a limit point of $E_0\}$, and $E_2 = E_1 \cup \{\omega\} \cup \{\beta\colon \exists\alpha \in E_1[\beta = \alpha^*]\}$. Let $\langle \alpha_\nu\colon \nu < \kappa_0\rangle$ be the continuous increasing enumeration of E_2, and let $\nu = \nu' + n$ for some $n \in \omega$. For β where $\beta \in [\alpha_\nu, \alpha_{\nu+1})$ in the first four cases, $\beta \in [\alpha_\nu^+, \alpha_{\nu+1})$ in the fifth case, and $\beta = \alpha_{\nu+1}$ in the last two cases, sets $C_i(\alpha_\nu, \beta)$ and $C_i(\alpha_\nu^+, \beta)$ are defined according to specific conditions on ν and ν' in the following manner:

1. $\nu' = \nu \ne 0$ and $n = 0$. Let then $p(\alpha_\nu)$ be the element p of R so that $p \cap \kappa_0 = \alpha_\nu$, and let $h_{p(\alpha_\nu)} : p(\alpha_\nu) \to \overline{p(\alpha_\nu)}$ be the order isomorphism between $p(\alpha_\nu)$ and $\overline{p(\alpha_\nu)}$. $C_1(\alpha_\nu, \beta) = \{h_{p(\alpha_\nu)}{}''p \cap \beta\colon p \in R_l,\ p \subseteq p(\alpha_\nu)$, and $h^{-1}_{p(\alpha_\nu)}(\beta) \in p\}$.
2. $\nu' \ne \nu$ and $n = 2k$. Let $C_2(\alpha_\nu, \beta) = \{h_{p(\alpha_\nu)}{}''p \cap \beta\colon p \in R$, and if $(\nu' \ne 0)$ or $(\nu' = 0$ and $k \ge 1)$, then $p(\alpha_{\nu'+2(k-1)}) \subset p \subseteq p(\alpha_\nu)\}$.
3. $\nu' \ne \nu$ and $n = 2k + 1$. Let $G(\alpha_\nu, \alpha_{\nu+1})$ be the projection of G onto $SC(\alpha_\nu, \alpha_{\nu+1})$. $C_3(\alpha_\nu, \beta) = \{p \cap \beta\colon \exists\pi \in G(\alpha_\nu, \alpha_{\nu+1})[p \in$p-part$(\pi)]\}$.

4. $n \neq 0$ or $\nu' = n = 0$. Let $H(\alpha_\nu, \alpha_{\nu+1})$ be the projection of G onto $\mathrm{Col}(\alpha_\nu, < \alpha_{\nu+1})$. $C_4(\alpha_\nu, \beta) = H(\alpha_\nu . \alpha_{\nu+1})|\beta$.
5. $n \neq 0$. Let $H(\alpha_\nu^+, \alpha_{\nu+1})$ be the projection of G onto $\mathrm{Col}(\alpha_\nu^+, < \alpha_{\nu+1})$. $C_5(\alpha_\nu^+, \beta) = H(\alpha_\nu^+, \alpha_{\nu+1})|\beta$.
6. $n \neq 0$ or $\nu' = n = 0$. With $H(\alpha_\nu, \alpha_{\nu+1})$ having the same meaning as in (4) above, $C_6(\alpha_\nu, \alpha_{\nu+1}) = H(\alpha_\nu, \alpha_{\nu+1})$.
7. $n \neq 0$. With $H(\alpha_\nu^+, \alpha_{\nu+1})$ having the same meaning as in (5) above,

$$C_7(\alpha_\nu^+, \alpha_{\nu+1}) = H(\alpha_\nu^+, \alpha_{\nu+1}).$$

We can now give a description of the model N_A witnessing the conclusions of our Theorem. Intuitively, N_A is V_{κ_0} of the least model of ZF extending V which contains, for β as above, $C_1(\alpha_\nu, \beta)$ if ν is a limit ordinal, $C_2(\alpha_\nu, \beta)$ if $\nu = \nu' + 2k$ and $\nu \in A$, $C_3(\alpha_\nu, \beta)$ if $\nu = \nu' + 2k + 1$ and $\nu \in A$, $C_4(\alpha_\nu, \beta)$ if $\nu \in B$, $\nu + 1 \in A$, and for $\nu - 1$ the immediate predecessor of ν (which exists since B is composed entirely of successor ordinals), $\nu - 1 \in B \cup \{0\}$, $C_5(\alpha_\nu^+, \beta)$ if $\nu \in B$, $\nu + 1 \in A$, and $\nu - 1 \in A$, $C_6(\alpha_\nu, \alpha_{\nu+1})$ if $\nu - 1, \nu, \nu + 1 \in B \cup \{0\}$, and $C_7(\alpha_\nu^+, \alpha_{\nu+1})$ if $\nu, \nu + 1 \in B$ and $\nu - 1 \in A$. The C_i have been chosen so as to ensure that successors of singular cardinals are measurable and successors of regular cardinals are non-measurable.

To define N_A more precisely, it is necessary to define canonical names $\underline{\alpha_\nu}$ for the α_ν's and canonical names $\underline{C_i(\nu, \beta)}$ and $\underline{C_i(\nu, \nu + 1)}$ for the seven sets just described. Recall that it is possible to decide $p(\alpha_\nu)$ (and hence $\overline{p(\alpha_\nu)}$) by writing $\omega \cdot \nu = \omega^{\sigma_0} \cdot n_0 + \omega^{\sigma_1} \cdot n_1 + \cdots + \omega^{\sigma_m} \cdot n_m$ (where $\sigma_0 > \sigma_1 > \cdots > \sigma_m$ are ordinals, $n_0, \ldots, n_m > 0$ are integers, and $+$, $\cdot$, and exponentiation are as in ordinal arithmetic), letting $\pi = \langle \langle p_{ij_i}, u_{ij_i}, C_{ij_i} \rangle_{i \leq m, 1 \leq j_i \leq n_i}, \langle \mu_{<\kappa_0^+}, C \rangle \rangle$ be so that $\min(p_{i1} \cap \kappa_0, \omega^{\mathrm{length}(u_{i1})}) = \sigma_i$ and $\mathrm{length}(u_{ij_i}) = \min(p_{i1} \cap \kappa_0, \mathrm{length}(u_{i1}))$ for $1 \leq j_i \leq n_i$, and letting $p(\alpha_\nu)$ be p_{mn_m}. Further, $D_\nu = \{r \in P\colon r|R_{<\kappa_0^+}$ extends a condition π of the above form$\}$ is a dense open subset of P. $\underline{\alpha_\nu}$ is the name of the α_ν determined by any element of $D_\nu \cap G$; in the notation of [G2], $\underline{\alpha_\nu} = \{\langle r, \check{\alpha}_\nu(r) \rangle\colon r \in D_\nu\}$, where $\alpha_\nu(r)$ is the α_ν determined by the condition r.

The canonical names $\underline{C_i(\nu, \beta)}$ and $\underline{C_i(\nu, \nu + 1)}$ are defined in a manner so as to be invariant under the appropriate group of automorphisms. Specifically, there are seven cases to consider. We again write $\nu = \nu' + n$ and let β be as before. We also assume without loss of generality that as in [G2], $\alpha_{\nu+1}$ is determined by D_ν. Further, we adopt throughout each of the seven cases the notation of [G2].

1. $\nu' \neq \nu \neq 0$ and $n = 0$. $\underline{C_1(\nu,\beta)} = \{\langle r, (\check{r}|R_{<\kappa_0^+})|(\alpha_\nu(r),\beta)\rangle : r \in D_\nu\}$, where for $r \in P$, $\pi =$ $r|R_{<\kappa_0^+}$, $\pi|(\alpha_\nu(r),\beta) = \{h_{p(\alpha_\nu)(r)}{}''p \cap \beta : p \in$p-part$(\pi)$, $p \subseteq p(\alpha_\nu)(r)$, $p \in R_l|\pi$, and $h^{-1}_{p(\alpha_\nu)(r)}(\beta) \in p$.
2. $\nu \in A$, $\nu' \neq \nu$, and $n = 2k$. Note that as in [G2] we can assume without loss of generality that for any $r \in D_\nu$, r determines $\alpha_{\nu'+2(k-1)}$. $\underline{C_2(\nu,\beta)} = \{\langle r, (\check{r}|R_{<\kappa_0^+})|(\alpha_\nu(r),\beta)\rangle : r \in D_\nu\}$, where this time for $r \in P$, $\pi = r|R_{<\kappa_0^+}$, $\pi|(\alpha_\nu(r),\beta) = \{h_{p(\alpha_\nu)(r)}{}''p \cap \beta : p \in$p-part$(\pi)$, $p \in R|\pi$, $p(\alpha_{\nu'+2(k-1)})(r) \subseteq p \subseteq p(\alpha_\nu)(r)$, and $h^{-1}_{p(\alpha_\nu)(r)}(\beta) \in p\}$.
3. $\nu \in A$, $\nu' \neq \nu$, and $n = 2k+1$.

$$\underline{C_3(\nu,\beta)} = \{\langle r, (\check{r}|SC(\alpha_\nu(r),\alpha_{\nu+1}(r)))|(\alpha_\nu(r),\beta)\rangle : r \in D_\nu\},$$

where for $r \in P$, $\pi = r|SC(\alpha_\nu(r),\alpha_{\nu+1}(r))$, $\pi|(\alpha_\nu(r),\beta) = \{p \cap \beta : p \in$p-part$(\pi)\}$.
4. $\nu - 1 \in B \cup \{0\}$, $\nu \in B$, and $\nu + 1 \in A$.

$$\underline{C_4(\nu,\beta)} = \{\langle r, (\check{r}|\mathrm{Col}(\alpha_\nu(r),\alpha_{\nu+1}(r)))|\beta\rangle : r \in D_\nu\}.$$

5. $\nu \in B, \nu-1, \nu+1 \in A$. $\underline{C_5(\nu,\beta)} = \{\langle r, (\check{r}|\mathrm{Col}(\alpha_\nu^+(r),\alpha_{\nu+1}(r)))|\beta\rangle : r \in D_\nu\}$.
6. $\nu - 1, \nu, \nu + 1 \in B \cup \{0\}$. $\underline{C_6(\nu,\nu+1)} = \{\langle r, (\check{r}|\mathrm{Col}(\alpha_\nu(r),\alpha_{\nu+1}(r)))\rangle : r \in D_\nu\}$.
7. $\nu, \nu+1 \in B$ and $\nu - 1 \in A$. $\underline{C_7(\nu,\nu+1)} = \{\langle r, (\check{r}|\mathrm{Col}(\alpha_\nu^+(r),\alpha_{\nu+1}(r)))\rangle : r \in D_\nu\}$.

As in [G2], since for any $r, r' \in D_\nu \cap G$, $p(\alpha_\nu)(r) = p(\alpha_\nu)(r')$, each of the definitions above is unambiguous.

Let $\mathcal{G}$ be the group of automorphisms of [G2], and let

$$\underline{C(G)} = \bigcup_{i=1}^{5} \{\pi(\underline{C_i(\nu,\beta)}) : \pi \in \mathcal{G}, \quad 0 < \nu < \kappa_0,$$

and $\beta \in [\nu, \kappa_0)$ is a cardinal$\} \cup \bigcup_{i=6}^{7} \{\pi(\underline{C_i(\nu,\nu+1)}) : \pi \in \mathcal{G}$ and $0 < \nu < \kappa_0\}$. $C(G) = \bigcup_{i=1}^{5} \{i_G(\pi(\underline{C_i(\nu,\beta)})) : \pi \in \mathcal{G}$, $0 < \nu < \kappa_0$, and $\beta \in [\nu,\kappa_0)$ is a cardinal$\} \cup \bigcup_{i=6}^{7} \{i_G(\pi(\underline{C_i(\nu,\nu+1)})) : \pi \in \mathcal{G}$ and $0 < \nu < \kappa_0\} = i_G(\underline{C(G)})$. N_A is then the set of all sets of rank $< \kappa_0$ of the model consisting of all sets which are hereditarily V definable from $C(G)$, i.e., $N_A = V_{\kappa_0}^{HVD(C(G))}$.

The arguments of [G2] show that $N_A \models ZF + \neg AC_\omega$. In addition, we know that for any ordinal γ and any set $x \subseteq \gamma$, $x \in N_A$, $x = \{\alpha < \gamma : V[G] \models$

$\varphi(\alpha, i_G(\pi_1(\underline{C_{i_1}(\nu_1,\beta_1)})),\ldots,i_G(\pi_n(\underline{C_{i_n}(\nu_n,\beta_n)})),C(G))\}$, where i_j is an integer, $1 \le j \le n$, $1 \le i_j \le 7$, each $\pi_i \in \mathcal{G}$, each β_i is an appropriate ordinal for i_j, and $\varphi(x_0,\ldots,x_{n+1})$ is a formula which may also contain some parameters from V which we shall suppress.

Let

$$\overline{P} = \prod_{\{i_j:\, i_j \in \{4,6\}, j \le n\}} \mathrm{Col}(\alpha_{\nu_j}, \beta_j) \times \prod_{\{i_j:\, i_j \in \{5,7\}, j \le n\}} \mathrm{Col}(\alpha^+_{\nu_j}, \beta_j)$$

$$\times \prod_{\{i_j:\, i_j = 3, j \le n\}} SC(\alpha_{\nu_j}, \beta_j) \times R_{<\kappa_0^+}.$$

For $\pi \in R_{<\kappa_0^+}$ and γ an arbitrary ordinal, let $\pi|\gamma = \{\langle q,u,C\rangle \in \pi\colon q \cap \kappa_0 \le \gamma\}$, and for $p \in \overline{P}$, $p = \langle p_1,\ldots,p_m,\pi\rangle$, $m \le n$, $\pi \in R_{<\kappa_0^+}$, let $p|\gamma = \langle q_1,\ldots,q_m,\pi|\gamma\rangle$, where $q_j = p_j$ if either α_{ν_j} or $\alpha^+_{\nu_j}$ is $\le \gamma$ and $q_j = \emptyset$ otherwise. In other words, $p|\gamma$ is the part of p below or at γ. Without loss of generality, we ignore the empty coordinates and let $\overline{P}|\gamma = \{p|\gamma\colon p \in \overline{P}\}$. Let $G|\gamma$ be the projection of G onto $\overline{P}|\gamma$. An analogous fact to Theorem 3.2.11 of [G2] holds, using the same proof as in [G2], namely for any $x \subseteq \gamma$, $x \in V[G|\gamma]$. In addition, the elements of $\overline{P}|\gamma$ can be partitioned into equivalence classes (the "almost similar" equivalence classes of [G2]) with respect to $\underline{C_{i_1}(\nu_1,\beta_1)},\ldots,\underline{C_{i_n}(\nu_n,\beta_n)}$ so that if $\sigma < \gamma$, τ is a term for x, and $p \Vdash \sigma \in \tau$, for any q in the same equivalence class as p, $q \Vdash \sigma \in \tau$. Further, if $\nu \in A$, then the arguments of [G2] show that for $\gamma = \alpha_{\nu+1}$ there are $< \alpha_{\nu+1}$ such equivalence classes. It is this last fact, in tandem with the way in which N_A was defined, that allows us to show that N_A is our desired model.

LEMMA 1: *$N_A \models$ "γ is a singular cardinal" iff $\gamma = \alpha_\nu$ for some $\nu \in A$.*

Proof of Lemma 1: In order to prove this lemma, we must first ascertain the nature of the cardinal structure of N_A. Specifically, we show that all (well-ordered) cardinals of N_A are either an α_ν or an $(\alpha^+_\nu)^V$ if $\nu = \sigma + 1$ and $\sigma \in A$. Thus, we begin by showing that any γ for $\gamma = \alpha_\nu$ or $\gamma = (\alpha^+_\nu)^V$ if $\nu = \sigma + 1$ and $\sigma \in A$ remains a cardinal in N_A.

Let γ be as just stated. If $x \subseteq \gamma$, $x \in N_A$, then as mentioned before, $x \in V[G|\gamma]$ where $G|\gamma$ is V-generic over $\overline{P}|\gamma$ for $\overline{P}|\gamma$, $G|\gamma$ as previously described. Thus, it suffices to show that γ remains a cardinal in $V[G|\gamma]$. To see this, observe that we can write $\overline{P}|\gamma$ as $Q_0 \times Q_1$, where Q_0 is a partial ordering (possibly trivial) defined over γ and some ordinal $\beta > \gamma$, and Q_1 is the rest of $\overline{P}|\gamma$. Since by the definition of N_A, Q_0 will be either trivial (if $\gamma = \alpha_\nu$ and $\nu - 1 \in A$), a partial ordering of the form $\mathrm{Col}(\gamma, < \beta)$, a partial ordering of the form $SC(\gamma, \beta)$, or a supercompact Radin forcing defined over $P_\gamma(\beta)$ isomorphic to a partial ordering of the form $SC(\gamma, \beta)$, forcing with Q_0 will preserve the fact that γ is a cardinal and preserve the same bounded subsets of γ as in V. Working now in V^{Q_0}, we can factor Q_1 as $Q_2 \times Q_3$, where Q_2 is a partial ordering defined over ordinals $\gamma' < \beta' \leq \gamma$ and Q_3 is the rest of Q_1. The fact that $\overline{P}|\gamma$ is a finite product allows us to assume that γ' and β' are the maximum such ordinals. Further, the fact that V and V^{Q_0} have the same bounded subsets of γ ensures that Q_2 can be partitioned into $< \gamma$ almost similar equivalence classes in both V and V^{Q_0} unless Q_2 is of the form $\mathrm{Col}(\gamma', < \gamma)$. In this case, the definition of N_A ensures that $V^{Q_0} \models$ "γ is a regular cardinal", so forcing over V^{Q_0} with Q_2 preserves the fact that γ is a regular cardinal; further, $V^{Q_0 \times Q_2} \models$ "Q_3 can be partitioned into $< \gamma$ almost similar equivalence classes". Thus, in either case, since $G|\gamma$ is V-generic over $\overline{P}|\gamma = Q_0 \times Q_1 = Q_0 \times Q_2 \times Q_3$, $V[G|\gamma] \models$ "γ is a cardinal".

Let now $\langle \beta_\nu \colon \nu < \kappa_0 \rangle$ be the continuous increasing enumeration of $\{\alpha_\nu \colon \nu < \kappa_0\} \cup \{(\alpha_\nu^+)^V \colon \nu = \sigma + 1$ and $\sigma \in A\}$. As in [G2], by the fact that the definition of N_A ensures that N_A contains collapse maps for each V cardinal in the interval $(\beta_\nu, \beta_{\nu+1})$ where $\nu < \kappa_0$ is arbitrary, it is inductively the case that $N_A \models$ "$\forall \nu [\beta_\nu \leq \aleph_\nu]$". Since each β_ν is a cardinal in N_A, $N_A \models$ "$\forall \nu [\beta_\nu = \aleph_\nu]$". Thus, the β_ν's and the cardinals of N_A coincide.

If $\gamma = \alpha_\nu$ for some $\nu \in A$, then since the definition of N_A ensures that N_A contains a cofinal ω sequence for α_ν, $N_A \models$ "α_ν is a singular cardinal". If $\gamma \neq \alpha_\nu$ for some $\nu \in A$ and $N_A \models$ "γ is a cardinal", then by the preceding paragraph, either $\gamma = \alpha_\nu$ for $\nu \in B$ or $\gamma = (\alpha_\nu^+)^V$ for $\nu = \sigma + 1$ and $\sigma \in A$. No matter which of these were true, if $x \in N_A$ coded a sequence witnessing the singularity of γ, then $x \in V[G|\gamma]$ for $G|\gamma$ as earlier. When factoring $\overline{P}|\gamma$ into $Q_0 \times Q_1$, since this case ensures Q_0 must be a Lévy collapse, $V^{Q_0} \models$ "γ is regular". Further, when factoring Q_1 into $Q_2 \times Q_3$, since our earlier discussion shows either Q_2 is of the form $\mathrm{Col}(\gamma', < \gamma)$ or is so that $V^{Q_1} \models$ "Q_2 can be partitioned into $< \gamma$ almost similar equivalence classes", $V^{Q_0 \times Q_2} \models$ "γ is regular". Therefore, since $V^{Q_0 \times Q_2} \models$ "Q_3 can be partitioned into $< \gamma$ almost similar equivalence classes" and $G|\gamma$ is V-generic over $Q_0 \times Q_2 \times Q_3$, $V[G|\gamma] \models$ "γ is regular". Thus, x cannot code a sequence witnessing the singularity of γ. This proves Lemma 1. ∎

LEMMA 2: *If $N_A \models$ "γ is the successor of a singular cardinal", then $N_A \models$ "γ is* *measurable via some normal measure".*

Proof of Lemma 2: By Lemma 1, for any γ as in the hypotheses, $N_A \models$ "$\gamma = \alpha_\nu^+$" where $\nu \in A$. Further, the definition of N_A ensures that $\gamma = \alpha_{\nu+1}$.

Fix $\mu \in V$ a normal measure over γ. In N_A, define $\mu^* = \{y \subseteq \gamma$: y contains a μ measure 1 set$\}$. We show $N_A \models$ "μ^* is a normal measure over γ". If $x \subseteq \gamma$, $x \in N_A$, then $x \in V[G|\gamma]$ for $G|\gamma$ V-generic over $\overline{P}|\gamma$, $\overline{P}|\gamma$, $G|\gamma$ as before. Further, as mentioned in the sentences immediately preceding the statement of Lemma 1, the elements of $\overline{P}|\gamma$ can be partitioned into $< \alpha_{\nu+1}$ many almost similar equivalence classes so that if p and q are in the same equivalence class, τ is a term for x, and p decides "$\sigma \in \tau$", then q decides "$\sigma \in \tau$" in the same way. Thus, as in the proof of Lemma 1.3 of [A3], in $V[G|\gamma]$, the Lévy-Solovay arguments [LS] show $\mu' = \{y \subseteq \gamma$: y contains a μ measure 1 set$\}$ is a normal measure over γ. In particular, since $x \in V[G|\gamma]$, either x or $\gamma - x$ will contain a μ measure 1 set. Further, if $N_A \models$ "$\langle x_\beta\colon \beta < \delta < \gamma\rangle$ is a sequence of μ^* measure 1 sets", then since $\langle x_\beta\colon \beta < \delta < \gamma\rangle$ can be coded by a single $x \subseteq \gamma$, for the appropriate $\overline{P}|\gamma$ and $G|\gamma$, both x and $\langle x_\beta\colon \beta < \delta < \gamma\rangle$ are elements of $V[G|\gamma]$. Thus, $V[G|\gamma] \models$ "$\bigcap_{\beta<\delta} x_\beta \in \mu'$", so $N_A \models$ "$\bigcap_{\beta<\delta} x_\beta \in \mu^*$". Finally, if $N_A \models$ "$f\colon \gamma \to \gamma$ is a regressive function", then since f can be coded by a set of ordinals, $f \in V[G|\gamma]$ for the appropriate $\overline{P}|\gamma$ and $G|\gamma$. Thus, $V[G|\gamma] \models$ "f is constant on a μ' measure 1 set", so $N_A \models$ "f is constant on a μ^* measure 1 set". This proves Lemma 2. ∎

LEMMA 3: *If $N_A \models$ "γ is not the successor of a singular cardinal", then $N_A \models$ "γ is not measurable".*

Proof of Lemma 3: By Lemma 1, for any γ as in the hypotheses, either $\gamma = (\alpha_\nu^+)^V$ for $\nu = \sigma + 1$ and $\sigma \in A$ or $\gamma = \alpha_\nu$ for some $\nu \in B$ so that $\nu - 1 \in B$. If $\gamma = (\alpha_\nu^+)^V$, then since $V \models AC$, V contains a sequence of length $(\alpha_\nu^+)^V$ of subsets of α_ν. Since $V \subseteq N_A$, this sequence is present in N_A also. It is well known (see [J2], Lemma 27.2, p. 298) that if such a sequence exists, regardless of whether AC is true, γ can't be measurable.

If $\gamma = \alpha_\nu$ for some $\nu \in B$ so that $\nu - 1 \in B$, then by the construction of N_A, N_A contains a set of the form $C_6(\alpha_{\nu-1}, \alpha_\nu)$ or a set of the form $C_7(\alpha_{\nu-1}^+, \alpha_\nu)$. Since $V[C_6(\alpha_{\nu-1}, \alpha_\nu)] \subseteq N_A$ or $V[C_7(\alpha_{\nu-1}^+, \alpha_\nu)] \subseteq N_A$, in either case, there will be present in N_A a sequence of subsets of some smaller cardinal of length α_ν. As in the last paragraph, the presence of such a sequence contradicts the measurability of γ. This proves Lemma 3. ∎

The above three lemmas complete the proof of our Theorem. ∎

Let us observe that the current state of forcing technology requires that if $N_A \models$ "α_ν is measurable", then $N_A \models$ "α_ν^+ is regular" (and of course, by our requirements, is non-measurable). If we wanted to have $N_A \models$ "α_ν is measurable and α_ν^+ is singular", then there would have to be some way to collapse a singular cardinal to be the successor of a measurable cardinal κ while preserving the measurability of κ. Unless κ is to become $\aleph_1$ (see [A1]), it is unknown how to do this. It is for this reason we require that N_A contains sets of the form $C_5(\alpha_\nu^+, \beta)$ and $C_7(\alpha_\nu^+, \alpha_{\nu+1})$, since their presence provides enough of a "gap" to ensure that α_ν remains measurable in N_A. More specifically, their presence ensures that in the analogue to Theorem 3.2.11 of [G2], since the Q_0 of Lemma 1 is trivial, the partial ordering $\overline{P}|\alpha_\nu$ can be partitioned into $< \alpha_\nu$ almost similar equivalence classes, the V-measurable cardinal which becomes $\alpha_{\nu-1}^+$ in N_A. This, as just shown, allows α_ν to remain measurable in N_A while preserving $(\alpha_\nu^+)^V$ as a regular cardinal.

In conclusion, we remark that from a weaker hypothesis than that assumed for our Theorem, i.e., from a cardinal κ_0 so that κ_0 is 2^λ supercompact for $\lambda > \kappa_0$ measurable, it is possible to construct a model for the theory "$ZF + \neg AC_\omega + \kappa > \omega$ is measurable iff κ is the successor of a singular cardinal". In this model, all limit cardinals will be singular and all successor cardinals will be regular, so the only measurable cardinals will be successors of limit cardinals. Thus, there is somewhat less flexibility as to what cardinals can be singular. An outline of the proof is as follows: If $j\colon V \to M$ witnesses that κ_0 is 2^λ supercompact for $\lambda > \kappa_0$ measurable, let κ_0^* in this case be the least measurable $> \kappa_0$, and let $R_{<\kappa_0^+}$ as before be supercompact Radin forcing over $P_{\kappa_0}(\kappa_0^*)$ defined using j. (The fact that κ_0 is 2^λ supercompact ensures this definition can be given.) Let

$$P' = R_{<\kappa_0^+} \times \prod_{\{\langle\alpha,\beta\rangle:\ \alpha<\beta<\kappa_0 \text{ are regular cardinals}\}} \mathrm{Col}(\alpha, <\beta),$$

and let P be the subordering consisting of all conditions of finite support. For $\langle \alpha_\nu : \nu < \kappa_0 \rangle$ as before, let N_A be V_{κ_0} of the least model of ZF extending V which contains the appropriate analogues of the sets $C_1(\alpha_\nu, \beta)$ if $\nu < \kappa_0$ is a limit ordinal and $\beta \in [\alpha_\nu, \alpha_{\nu+1})$, $C_7(\alpha_\nu^+, \alpha_{\nu+1})$ if ν is the successor of a limit ordinal, and $C_6(\alpha_\nu, \alpha_{\nu+1})$ if ν is neither a limit ordinal nor the successor of a limit ordinal. N_A can then be shown to be our desired model. Further, as the referee has pointed out, the methods of this paper can be used to construct, from the hypotheses of this paragraph, a model for the theory "$ZF + \neg AC_\omega$ + For every ordinal α, either $\aleph_\alpha$ or $\aleph_{\alpha+1}$ is measurable", i.e., a model in which every second cardinal is measurable.

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